



# A Schrödinger singular perturbation problem

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## Abstract

Consider the equation  $-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon)$  in  $\mathbb{R}^3$ ,  $|u(\infty)| < \infty$ ,  $\varepsilon = \text{const} > 0$ . Under what assumptions on  $q(x)$  and  $f(u)$  can one prove that the solution  $u_\varepsilon$  exists and  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon = u(x)$ , where  $u(x)$  solves the limiting problem  $q(x)u = f(u)$ ? These are the questions discussed in the paper.

## 1 Introduction

Let

$$-\varepsilon^2 \Delta u_\varepsilon + q(x)u_\varepsilon = f(u_\varepsilon) \text{ in } \mathbb{R}^3, \quad |u_\varepsilon(\infty)| < \infty, \quad (1.1)$$

$\varepsilon = \text{const} > 0$ ,  $f$  is a nonlinear smooth function,  $q(x) \in C(\mathbb{R}^3)$  is a real-valued function

$$a^2 \leq q(x), \quad a = \text{const} > 0. \quad (1.2)$$

We are interested in the following questions:

- 1) Under what assumptions does problem (1.1) have a solution?
- 2) When does  $u_\varepsilon$  converge to  $u$  as  $\varepsilon \rightarrow 0$ ?

Here  $u$  is a solution to

$$q(x)u = f(u). \quad (1.3)$$

The following is an answer to the first question.

**Theorem 1.1.** *Assume  $q \in C(\mathbb{R}^3)$ , (1.2) holds,  $f(0) \neq 0$ , and  $a$  is sufficiently large. More precisely, let  $M(R) := \max_{|u| \leq R} |f(u)|$ ,  $M_1(R) = \max_{|\xi| \leq R} |f'(\xi)|$ ,  $p := q(x) - a^2$ , and assume that  $\frac{\|p\|_{R+M(R)}}{a^2} \leq R$ , and  $\frac{\|p\| + M_1(R)}{a^2} \leq \gamma < 1$ , where  $\gamma > 0$  is a constant and  $\|p\| := \sup_{x \in \mathbb{R}^3} |p(x)|$ . Then equation (1.1) has a solution  $u_\varepsilon \not\equiv 0$ ,  $u_\varepsilon \in C(\mathbb{R}^3)$ , for any  $\varepsilon > 0$ .*

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In Section 4 the potential  $q$  is allowed to grow at infinity.

An answer to the second question is:

**Theorem 1.2.** *If  $\frac{f(u)}{u}$  is a monotone, growing function on the interval  $[u_0, \infty)$ , such that  $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty$ , and  $\frac{f(u_0)}{u_0} < a^2$ , where  $u_0 > 0$  is a fixed number, then there is a solution  $u_\varepsilon$  to (1.1) such that*

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x) = u(x), \quad (1.4)$$

where  $u(x)$  solves (1.3).

Singular perturbation problems have been discussed in the literature [1], [3], [5], but our results are new.

In Section 2 proofs are given.

In Section 3 an alternative approach is proposed.

In Section 4 an extension of the results to a larger class of potentials is given.

## 2 Proofs

*Proof of Theorem 1.1.* The existence of a solution to (1.1) is proved by means of the contraction mapping principle.

Let  $g$  be the Green function

$$(-\varepsilon^2 \Delta + a^2)g = \delta(x - y) \text{ in } \mathbb{R}^3, \quad g := g_a(x, y, \varepsilon) \xrightarrow{|x| \rightarrow \infty} 0, \quad g = \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2}. \quad (2.1)$$

Let  $p := q - a^2 \geq 0$ . Then (1.1) can be written as:

$$u_\varepsilon(x) = - \int_{\mathbb{R}^3} g p u_\varepsilon dy + \int_{\mathbb{R}^3} g f(u_\varepsilon) dy := T(u_\varepsilon). \quad (2.2)$$

Let  $X = C(\mathbb{R}^3)$  be the Banach space of continuous and globally bounded functions with the sup-norm:  $\|v\| := \sup_{x \in \mathbb{R}^3} |v(x)|$ . Let  $B_R := \{v : \|v\| \leq R\}$ .

We choose  $R$  such that

$$T(B_R) \subset B_R \quad (2.3)$$

and

$$\|T(v) - T(w)\| \leq \gamma \|v - w\|, \quad v, w \in B_R, \quad 0 < \gamma < 1. \quad (2.4)$$

If (2.3) and (2.4) hold, then the contraction mapping principle yields a unique solution  $u_\varepsilon \in B_R$  to (2.2), and  $u_\varepsilon$  solves problem (1.1).

The assumption  $f(0) \neq 0$  guarantees that  $u_\varepsilon \not\equiv 0$ .

Let us check (2.3). If  $\|v\| \leq R$ , then

$$\|T(v)\| \leq \|v\| \|p\| \int_{\mathbb{R}^3} g(x, y) dy + \frac{M(R)}{a^2} \leq \frac{\|p\| R + M(R)}{a^2}, \quad (2.5)$$

where  $M(R) := \max_{|u| \leq R} |f(u)|$ . Here we have used the following estimate:

$$\int_{\mathbb{R}^3} g(x, y) dy = \int_{\mathbb{R}^3} \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2} dy = \frac{1}{a^2}. \quad (2.6)$$

If  $\|p\| < \infty$  and  $a$  is such that

$$\frac{\|p\|R + M(R)}{a^2} \leq R, \quad (2.7)$$

then (2.3) holds.

Let us check (2.4). Assume that  $v, w \in B_R$ ,  $v - w := z$ . Then

$$\|T(v) - T(w)\| \leq \frac{\|p\|}{a^2} \|z\| + \frac{M_1(R)}{a^2} \|z\|, \quad (2.8)$$

where, by the Lagrange formula,  $M_1(R) = \max_{|\xi| \leq R} |f'(\xi)|$ . If

$$\frac{\|p\| + M_1(R)}{a^2} \leq \gamma < 1, \quad (2.9)$$

then (2.4) holds. By the contraction mapping principle, (2.7) and (2.9) imply the existence and uniqueness of the solution  $u_\varepsilon(x)$  to (1.1) in  $B_R$  for any  $\varepsilon > 0$ .

Theorem 1.1 is proved.  $\square$

*Proof of Theorem 1.2.* In the proof of Theorem 1.1 the parameters  $R$  and  $\gamma$  are independent of  $\varepsilon > 0$ . Let us denote by  $T_\varepsilon$  the operator defined in (2.2). Then

$$\lim_{\varepsilon \rightarrow 0} \|T_\varepsilon(v) - T_0(v)\| = 0, \quad (2.10)$$

for every  $v \in C(\mathbb{R}^3)$ , where the limiting operator  $T_0$ , corresponding to the value  $\varepsilon = 0$ , is of the form:

$$T_0(v) = \frac{-pv + f(v)}{a^2}. \quad (2.11)$$

To calculate  $T_0(v)$  we have used the following formula

$$\lim_{\varepsilon \rightarrow 0} g_a(x, y, \varepsilon) = \frac{1}{a^2} \delta(x - y), \quad (2.12)$$

where convergence is understood in the following sense: for every  $h \in C(\mathbb{R}^3)$  one has:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} g_a(x, y, \varepsilon) h(y) dy = \frac{h(x)}{a^2}, \quad a > 0. \quad (2.13)$$

Indeed, one can easily check that

$$\lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq c > 0} g_a(x, y, \varepsilon) dy = 0, \quad \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \leq c} g_a(x, y, \varepsilon) dy = \frac{1}{a^2}, \quad a > 0, \quad (2.14)$$

where  $c > 0$  is an arbitrary small constant. These two relations imply (2.13).

We *claim* that if (2.10) holds for every  $v \in X$ , and  $\gamma$  in (2.4) does not depend on  $\varepsilon$ , then (1.4) holds, where  $u$  solves the limiting equation (2.2):

$$u = T_0(u) = \frac{-pu + f(u)}{a^2}. \quad (2.15)$$

Equation (2.15) is equivalent to (1.3). The assumptions of Theorem 1.2 imply that equation (1.3) has a unique solution.

Let us now prove the above *claim*.

Let  $u = T_\varepsilon(u)$ ,  $u = u_\varepsilon$ ,  $v = T_0(v)$ , and  $\lim_{\varepsilon \rightarrow 0} \|T_\varepsilon(w) - T_0(w)\| = 0$  for all  $w \in X$ . Assume that  $\|T_\varepsilon(v) - T_\varepsilon(w)\| \leq \gamma \|v - w\|$ ,  $0 < \gamma < 1$ , where the constant  $\gamma$  does not depend on  $\varepsilon$ , so that  $T_\varepsilon$  is a contraction map. Consider the iterative process  $u_{n+1} = T_\varepsilon(u_n)$ ,  $u_0 = v$ . The usual estimate for the elements  $u_n$  is:  $\|u_n - v\| \leq \frac{1}{1-\gamma} \|T_\varepsilon v - v\|$ . Let  $u = \lim_{n \rightarrow \infty} u_n$ . This limit does exist because  $T_\varepsilon$  is a contraction map. Taking  $n \rightarrow \infty$ , one gets  $\|u - v\| \leq \frac{1}{1-\gamma} \|T_\varepsilon(v) - T_0(v)\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The *claim* is proved.

Theorem 1.2 is proved.  $\square$

**Remark 2.1.** *Conditions of Theorem 1.1 and of Theorem 1.2 are satisfied if, for example,  $q(x) = a^2 + 1 + \sin(\omega x)$ , where  $\omega = \text{const} > 0$ ,  $f(u) = (u + 1)^m$ ,  $m > 1$ , or  $f(u) = e^u$ . If  $R = 1$ , and  $f(u) = e^u$ , then  $M(R) = e$ ,  $M_1(R) = e$ ,  $\|p\| \leq 2$ , so  $\frac{2+e}{a^2} \leq 1$  and  $\frac{2+e}{a^2} \leq \gamma < 1$  provided that  $a > \sqrt{5}$ . For these  $a$ , the conditions of Theorem 1.1 are satisfied and there is a solution to problem (1.1) in the ball  $B_1$  for any  $\varepsilon > 0$ .*

### 3 A different approach

Let us outline a different approach to problem (1.1). Set  $x = \xi + \varepsilon y$ . Then

$$-\Delta_y w_\varepsilon + a^2 w_\varepsilon + p(\varepsilon y + \xi) w_\varepsilon = f(w_\varepsilon), \quad |w_\varepsilon(\infty)| < \infty, \quad (3.1)$$

$w_\varepsilon := u_\varepsilon(\varepsilon y + \xi)$ ,  $p := q(\varepsilon y + \xi) - a^2 \geq 0$ . Thus

$$w_\varepsilon = - \int_{\mathbb{R}^3} G(x, y) p(\varepsilon y + \xi) w_\varepsilon dy + \int_{\mathbb{R}^3} G(x, y) f(w_\varepsilon) dy, \quad (3.2)$$

where

$$(-\Delta + a^2)G = \delta(x - y) \text{ in } \mathbb{R}^3, \quad G = \frac{e^{-a|x-y|}}{4\pi|x-y|}, \quad a > 0. \quad (3.3)$$

One has

$$\int_{\mathbb{R}^3} G(x, y) dy = \frac{1}{a^2}. \quad (3.4)$$

Using an argument similar to the one in the proofs of Theorem 1.1 and Theorem 1.2, one concludes that for any  $\varepsilon > 0$  and any sufficiently large  $a$ , problem (3.1) has a unique

solution  $w_\varepsilon = w_\varepsilon(y, \xi)$ , which tends to a limit  $w = w(y, \xi)$  as  $\varepsilon \rightarrow 0$ , where  $w$  solves the limiting problem

$$-\Delta_y w + q(\xi)w = f(w), \quad |w(\infty, \xi)| < \infty. \quad (3.5)$$

Problem (3.5) has a solution  $w = w(\xi)$ , which is indepnd of  $y$  and solves the equation

$$q(\xi)w = f(w). \quad (3.6)$$

The solution to (3.5), bounded at infinity, is unique if  $a$  is sufficiently large. This is proved similarly to the proof of (2.9). Namely, let  $b^2 := q(\xi)$ . Note that  $b \geq a$ . If there are two solutions to (3.5), say  $w$  and  $v$ , and if  $z := w - v$ , then  $\|z\| \leq b^{-2}M_1(R)\|z\| < \|z\|$ , provided that  $b^{-2}M_1(R) < 1$ . Thus  $z = 0$ , and the uniqueness of the solution to (3.5) is proved under the assumption  $q(\xi) > M_1(R)$ , where  $M_1(R) = \max_{|\xi| \leq R} |f'(\xi)|$ .

Replacing  $\xi$  by  $x$  in (3.6), we obtain the solution found in Theorem 1.2.

## 4 Extension of the results to a larger class of potentials

Here a method for a study of problem (1.1) for a larger class of potentials  $q(x)$  is given. We assume that  $q(x) \geq a^2$  and can grow to infinity as  $|x| \rightarrow \infty$ . Note that in Sections 1 and 2 the potential was assumed to be a bounded function. Let  $g_\varepsilon$  be the Green function

$$-\varepsilon^2 \Delta_x g_\varepsilon + q(x)g_\varepsilon = \delta(x - y) \text{ in } \mathbb{R}^3, \quad |g_\varepsilon(\infty, y)| < \infty. \quad (4.1)$$

As in Section 2, problem (1.1) is equivalent to

$$u_\varepsilon = \int_{\mathbb{R}^3} g_\varepsilon(x, y) f(u_\varepsilon(y)) dy, \quad (4.2)$$

and this equation has a unique solution in  $B_R$  if  $a^2$  is sufficiently large. The proof, similar to the one given in Section 2, requires the estimate

$$\int_{\mathbb{R}^3} g_\varepsilon(x, y) dy \leq \frac{1}{a^2}. \quad (4.3)$$

Let us prove inequality (4.3). Let  $G_j$  be the Green function satisfying equation (4.1) with  $q = q_j$ ,  $j = 1, 2$ . Estimate (4.3) follows from the inequality

$$G_1 \leq G_2 \quad \text{if } q_1 \geq q_2. \quad (4.4)$$

This inequality can be derived from the maximum principle.

If  $q_2 = a^2$ , then  $G_2 = \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2}$ , and the inequality  $g_\varepsilon(x, y) \leq \frac{e^{-\frac{a}{\varepsilon}|x-y|}}{4\pi|x-y|\varepsilon^2}$  implies (4.3).

Let us prove the following relation:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} g_\varepsilon(x, y) h(y) dy = \frac{h(x)}{q(x)} \quad \forall h \in \mathring{C}^\infty(\mathbb{R}^3), \quad (4.5)$$

where  $\overset{\circ}{C}^\infty$  is the set of  $C^\infty(\mathbb{R}^3)$  functions vanishing at infinity together with their derivatives. This formula is an analog to (2.12).

To prove (4.5), multiply (4.1) by  $h(y)$ , integrate over  $\mathbb{R}^3$  with respect to  $y$ , and then let  $\varepsilon \rightarrow 0$ . The result is (4.5). More detailed argument is given at the end of the paper.

Thus, Theorem 1.1 and Theorem 1.2 remain valid for  $q(x) \geq a^2$ ,  $a > 0$  sufficiently large, provided that  $\frac{f(u)}{u}$  monotonically growing to infinity and  $\frac{f(u_0)}{u_0} < a^2$  for some  $u_0 > 0$ . Under these assumptions the solution  $u(x)$  to the limiting equation (1.3) is the limit of the solution to (4.2) as  $\varepsilon \rightarrow 0$ .

Let us give details of the proof of (4.5). Denote the integral on the left-hand side of (4.5) by  $w = w_\varepsilon(x)$ . From (4.1) it follows that

$$-\varepsilon^2 \Delta w_\varepsilon + q(x)w_\varepsilon = h(x). \quad (4.6)$$

Multiplying (4.6) by  $w_\varepsilon$  and integrating by parts yields the estimate  $\|w_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq c$ , where  $c > 0$  is a constant independent of  $\varepsilon$ . Consequently, one may assume that  $w_\varepsilon$  converges weakly in  $L^2(\mathbb{R}^3)$  to an element  $w$ . Multiplying (4.6) by an arbitrary function  $\phi \in C_0^\infty(\mathbb{R}^3)$ , integrating over  $\mathbb{R}^3$ , then integrating by parts the first term twice, and then taking  $\varepsilon \rightarrow 0$ , one obtains the relation:

$$\int_{\mathbb{R}^3} q(x)w(x)\phi(x)dx = \int_{\mathbb{R}^3} h(x)\phi(x)dx, \quad (4.7)$$

which holds for all  $\phi \in C_0^\infty(\mathbb{R}^3)$ . It follows from (4.7) that  $qw = h$ . This proves formula (4.5).

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